OCU-E Discussion Paper Series

Uzawa, Solow, and Aggregation

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September 29, 2022

Discussion Paper No.008

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Abstract

The objective of the present paper is to aggregate Uzawa's two-sector economy into Solow's one-sector economy. We give a definition of aggregating economy and establish aggregation possibility theorems. A major problem in this regard is how to obtain an aggregate production function and an aggregate commodity price. We present them in an explicit way. Furthermore, we present a maximization problem that is equivalent with the temporary equilibria of Uzawa's two-sector model to prove that the defined aggregate production function is monotonous, linearly homogenous and concave.

Key words: aggregation, two-sector economy, one-sector economy

JEL classifications: E13, E23, D21, O41

*This study is supported by a Grant-in-Aid for Scientific Research (grant number 19K01629). We are much indebted to Professors Hiroaki Nagatani(Osaka University) and Norio Takeoka(Hitotsubashi University) for helpful comments and discussions.

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1 Introduction

The real economy is very much complicated. Economists hardly take hold of whole economy itself. They usually have recourse to economic models to study the behavior of an economy. They make use of n-consumers and m-producers general equilibrium model, of two-sector growth model or of one-sector macro economic model according to their objectives.

Economists, however, have hardly often scrutinized relations between models. For example, we do not know the way how we reduce Uzawa's two-sector model to Solow's one-sector model (see Uzawa (1963) and Solow (1956)). It is sometimes said that the two-sector model is a general version of one-sector model. Is this intuition right? Our question is here. That is, we say that two-sector model is more general than one-sector model safely when we can aggregate two sector-model to one-sector model in a consistent way. We face with two problems in resolving this question. (P1) We need to find an aggregate production function for aggregated commodity. (P2) We need to have a price of aggregated commodity.

The problem (P1) to obtain an aggregate production function is classical and now abound in literature known as Cambridge Controversy, which are surveyed and summarized in Felipe and Fisher (2003) and Fisher (1993). In Cambridge Controversy, they have basic common plan to construct an aggregate production function from purely informations on production technologies. As a result of long period of discussions, we are forced to stand no chance of making use of the aggregate production function. That is, very limited class of production functions, e.g., a class consisting of an identical function, can be consistent with aggregate production function. Finally, Felipe and McCombie (2013) depicts this situation as that the aggregate production function is 'not even wrong'.

On the other hand, Baquee and Farhi (2019) presents an aggregate production function from a new angle, that is, from the allcational efficiency point of view. Their way of constructing aggregate production function follows next steps:

(step 1) Assume that there exists an "aggregator", a non-negative valued function whose variables are total consumption goods.

(step 2) Maximize the value of aggregator under the resourse constraints to find an efficient allocation in an aggregative sense.

(step 3) The maximizing solution consists of the quantities of consumption goods, each of which is a function of the quantity of primary factors of production, i.e., capitals and labors.

(step 4) Finally, substituting obtained functions into the aggregator gives an aggregate production function.

This method is successful in the sense that it permits the wide varieties of production technologies. Baquee and Farhi (2019), however, remains some questions unanswered. First, they do not give a concrete functional form of 'Aggregator'. Second, they do not give the price of aggregate product. This is nothing but the problem (P2) above.

Recently, to these problems, Doi, Fujii, Horie, Iritani, Sato and Yasuoka (2021) gives an affirmative answer by way of Cobb-Douglas example. They aggregate two-sector economy into one-sector economy in a temporary equilibrium setting. They show aggregate possibility results which are summarized as follows.

R1 The aggregate production function takes the form of $\prod_{i=1}^{2} (F_i/\alpha_i)^{\alpha_i}$, when $F_i, i = 1, 2$ are production functions of sectors. The coefficients α_1 and α_2 are assumed to be constants which represent the ratios of spending on the i-th commodity to total income of the two-sector economy.

R2 The price of aggregated commodity is represented as a geometric mean of individual prices.

We extend the model with restrictive Cobb-Douglas production functions by Doi, Fujii, Horie, Iritani, Sato and Yasuoka (2021) into that with general neoclassical production functions. In the present paper, we concentrate ourself to the temporary equilibrium since we want to treat an aggregation problem as simple as possible. We present a definition of aggregating economy and establish two aggregation possibility theorems. One theorem is for the existence of the aggregated commodity price and that of the aggregate production function. The other theorem is for the aggregation Uzawa-economy to Solow-economy.

The present paper consists of two parts. In the first part, we define the aggregation of two-sector economy into one-sector economy and search a candidate of aggregate price and that of aggregate production function. This part is given in Section 3. In the second part, we show that obtained candidates are those that we want to get. In this step, we make most use of the contribution by Baqaee and Farhi (2019). The key concept of Baqaee and Farhi (2019) is the Aggregator function. We employ as an Aggregator function the candidate of aggregate production function obtained in the first part. This bring in three important results. One is that we can describe Uzawa's temporary equilibrium as a simple maximization problem. Another is that the Uzawa's temporary equilibrium turns out to be unique. The other is that we can establish aggregation possibility theorems. We emphasize that the results 1 and

2 above still hold in the general setting. These are characteristic features of our possibility theorems. The second part is given in Sections 4 and 5.

We propose a new definition of an aggregate production function in this paper. Suppose that the exists a production function under which the total demand for factors are the solution to the profit maximization problem. We call this production function as an aggregate production function. We will give a precise definition of aggregation in section 3.1 and prove possibility of aggregation in Section 5.

2 Two-Sector Model

In this section, we introduce a basic notations and concepts in the present paper.

2.1 Notations, Assumptions and Uzawa-economy

Our starting point is a temporary equilibrium in two-sector model presented by Uzawa (1963). The producer i, or the i-th sector, produces product Y_i by employing capital K_i and labor L_i, i=1,2. A function F_i denote the i-th sector's production function. That is[1],

$$Y_i = F_i(K_i, L_i),\ (K_i, L_i) \in \mathbb{R}^2_+,\ i = 1, 2.$$

Assumption 1 *We assume each production function F_i satisfies following conditions* A1, A2, A3, *and* A4, $i = 1, 2$.

A1 *Each production function F_i is twice continuously differentiable, concave and homogenous of degree one.*

By homogeneity of F_i, we can represent per capita production function by $f_i(k_i)$, where $f_i(k_i) = F_i(k_i, 1)$ and $k_i = K_i/L_i$.

A2 *For any positive K_i, L_i, it holds that $F_i(0, L_i) = F_i(K_i, 0) = 0$, $i = 1, 2$.*

A3 *F_i is twice continuously differentiable with respect to $(K_i, L_i) \in \mathbb{R}^2_{++}$ and satisfies:*

$$\frac{\partial F_i}{\partial L_i} > 0,\ \frac{\partial F_i}{\partial K_i} > 0,\ \frac{\partial^2 F_i}{\partial {L_i}^2} < 0,\ \frac{\partial^2 F_i}{\partial {K_i}^2} < 0,\ i = 1, 2.$$

[1]The symbols $\mathbb{R}^2_+$ and $\mathbb{R}^2_{++}$ represent a nonnegative orthant and a positive orthant in two dimensional Euclid space $\mathbb{R}^2$, respectively.

A4 *F_i is well-behaved, that is, has a following property:*

$$\lim_{k_i \to 0} f_i{}'(k_i) = \infty \text{ and } \lim_{k_i \to \infty} f_i{}'(k_i) = 0, \ i = 1, 2.$$

The following property on a production function hold.

Lemma 1 *Suppose that a production function $F_i(K_i, L_i)$ satisfies* (A1)–(A4)*, then $F_i(K_i, L_i)$ is strictly concave without along the path from the origin in the domain.*

The proof of Lemma 1 is relegated to Mathematical Appendix.

We model the demand side of the economy as simply as possible after Uzawa (1963). That is, we assume the amount of the expenditure on commodity i is proportional to income. The rates are represented by two positive constants α_1 and α_2 satisfying $\alpha_1 + \alpha_2 = 1$. We call these expenditure coefficients. The demand for commodity i is $\alpha_i \times I$, when I is a national income. The primary factors, capital K and labor L are endowed with an economy. Of course, K and L are positive. Let F_i be a production function of the sector i satisfying Assumption 1, $i = 1, 2$. A triplet $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ is a Uzawa-economy. In what follows, we measure prices by the rental rate of capital. Let ω and $p_i, i = 1, 2$ be the wage rental ratio and the commodity i's price rental ratio, respectively.

Definition 1 (Uzawa's Temporary Equilibrium) *Suppose that a Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ is given. A pair of a price vector and an allocation $((\omega^*, (p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ is the Uzawa temporary equilibrium if and only if the pair satisfies following two conditions* (U1) *and* (U2)*.*

(U1) *(Y_i^*, K_i^*, L_i^*) is a solution to the problem:*

$$\max_{Y_i, K_i, L_i} p_i^* Y_i - K_i - \omega^* L_i \quad \text{subject to } Y_i = F_i(K_i, L_i), \ i = 1, 2. \tag{1}$$

(U2) *Factors and commodities markets are in balance.*

$$\text{equilibrium of primary factors} \quad \begin{cases} K_1^* + K_2^* = K \\ L_1^* + L_2^* = L, \end{cases} \tag{2}$$

$$\text{equilibrium of commodities} \quad \begin{cases} \alpha_1(K + \omega^* L) = p_1^* Y_1^* \\ \alpha_2(K + \omega^* L) = p_2^* Y_2^*. \end{cases} \tag{3}$$

The condition (U1) implies that commodity i's supply Y_i^* and demands for factors K_i^*, L_i^* are determined by firm's profit maximization behavior, $i = 1, 2$. The equation (2) in (U2) implies that each factor market is in balance. The amount of demand for the commodity i is $\alpha_i(K + \omega^* L)$ and the amount of supply is $p_i^* F_i(K_i^*.L_i^*)$. Then (3) implies that the market of commodity i is in equilibrium.

2.2 Profit Maximization and Marginal Conditions

Arranging marginal conditions for profit maximization of the i-th sector, we have

$$\omega = \frac{f_i(k_i)}{f_i{}'(k_i)} - k_i. \tag{4}$$

Denote the right hand side of (4) by a function $\phi_i(k_i)$ and we have

$$\lim_{k_i \to 0} \phi_i(k_i) = 0, \ \lim_{k_i \to \infty} \phi_i(k_i) = \infty,$$

since the production function is well behaved. Furthermore, it holds that

$$\phi_i{}'(k_i) = -\frac{f_i(k_i) f_i{}''(k_i)}{f_i{}'(k_i)^2} > 0.$$

This implies that ϕ_i is strictly increasing and its infimum is zero and its supremum is infinity. We have unique capital-labor ratio corresponding to a wage rental ratio $\omega > 0$. This functional relation is expressed by a function $k_i(\omega)$, which satisfies

$$\lim_{\omega \to \infty} k_i(\omega) = \infty, \ \lim_{\omega \to 0} k_i(\omega) = 0. \tag{5}$$

The implicit function theorem assures us of the differentiability of $k_i(\omega)$ with respect to ω. And thus we know:

$$k_i{}'(\omega) = \frac{1}{\phi_i{}'(k_i(\omega))} > 0, i = 1, 2. \tag{6}$$

The equation (4) implies:

$$p_i(\omega) = \frac{1}{f_i{}'(k_i(\omega))}, \quad \omega = \frac{f_i(k_i(\omega))}{f_i'(k_i(\omega))} - k_i(\omega).$$

The first equality implies that ω determines p_i. The second equality is an identity which is rearranged again as follows:

$$\frac{1}{\omega + k_i(\omega)} = \frac{f_i'(k_i(\omega))}{f_i(k_i(\omega))}.$$

Multiplying $1 + k_i'(\omega)$ to both sides, we know:

$$\frac{1 + k_i'(\omega)}{\omega + k_i(\omega)} = \frac{f_i'(k_i(\omega))}{f_i(k_i(\omega))} + \frac{f_i'(k_i(\omega))}{f_i(k_i(\omega))} k_i'(\omega).$$

This is a differential equation. And thus we have:

$$\int \frac{1}{\omega + k_i(\omega)} \mathrm{d}\omega = \int \frac{f_i'(k_i(\omega))}{f_i(k_i(\omega))} \mathrm{d}\omega = \log(\omega + k_i(\omega)) - \log f_i(k_i(\omega)) + C_i, \tag{7}$$

where C_i is a constant. The equation (7) will turn out to be a key by which we can find a candidate of an aggregate production function.

2.3 Temporary Equilibrium

Functions $k_i(\omega), p_i(\omega), i = 1, 2$ are those defined in the previous sections. Let us define factor demands by use of equilibrium condition of commodities (3). Let ω be a positive real. We determine $L_i(\omega), i = 1, 2$ so as to satisfy next equation:

$$\alpha_i(k + \omega)L = p_i(\omega) f_i(k_i(\omega)) L_i(\omega),\ \ i = 1, 2, \tag{3$'$}$$

where $k = K/L$. By marginal condition for maximization, we can eliminate the term $p_i(\omega)$ in (3′) and rewrite (3′) to

$$\alpha_i(k + \omega)L = \frac{f_i(k_i(\omega))L_i(\omega)}{f_i'(k_i(\omega))} = (\omega + k_i(\omega))L_i(\omega),\ \ i = 1, 2.$$

And then, $K_1(\omega)$ and $K_2(\omega)$ are determined by $L_1(\omega)$ and $L_2(\omega)$. That is,

$$L_i(\omega) = \alpha_i \frac{k + \omega}{k_i(\omega) + \omega} L,\ \ i = 1, 2, \tag{8}$$

$$K_i(\omega) = k_i(\omega) L_i(\omega),\ \ i = 1, 2. \tag{9}$$

Now, we have $((\omega, (p_i(\omega))_{i=1}^2), (K_i(\omega), L_i(\omega))_{i=1}^2)$ as functions of ω. Finally, from (8) and (9), we have an identity:

$$\sum_{i=1}^{2} (K_i(\omega) + \omega L_i(\omega)) = K + \omega L. \tag{10}$$

This is the Walras' law. That is, the equilibrium of labor market implies that of capital market.

Now, let us define

$$\rho_i(\omega) = \frac{L_i(\omega)}{L} = \alpha_i \frac{k+\omega}{k_i(\omega)+\omega}, i = 1, 2. \tag{11}$$

The equilibrium in labor market is represented by an equation:

$$\rho_1(\omega) + \rho_2(\omega) = 1.$$

We know the facts that $k_i(\omega)$ is strictly increasing and has the zero infimum and the infinite supremum. Note that k in numerator of (11) is positive constant. Then we have $\rho_i(\underline{\omega}) > \alpha_i, i = 1, 2$ for a sufficiently small $\underline{\omega}$ and have $\rho_i(\bar{\omega}) < \alpha_i, i = 1, 2$ for a sufficiently large $\bar{\omega}$. The intermediate-value theorem assures us of the existence of ω^* which equates the demand and the supply of labor. This shows the existence of temporary equilibrium of Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$.

Let a pair $((\omega^*, (p_i(\omega^*))_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be a Uzawa temporary equilibrium. Then, we have following interesting equalities in equilibrium.

$$\begin{aligned} p_1(\omega^*)^{\alpha_1} p_2(\omega^*)^{\alpha_2} & \left(\frac{F_1(K_1^*, L_1^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2^*, L_2^*)}{\alpha_2}\right)^{\alpha_2} \\ & = K + \omega^* L \\ & = p_1(\omega^*) F_1(K_1^*, L_1^*) + p_2(\omega^*) F_2(K_2^*, L_2^*). \end{aligned} \tag{12}$$

The equality (12) will play an important role in Section 5.

2.4 A Sufficient Condition for Comparative Statics

We are going to prepare a condition by use of which we can describe responses of endogenous variables when exogenous parameters (K, L) vary. The condition will support the differentiability of endogenous variables. The wage rental ratio ω^* which equilibrates the labor markets is the solution to the following equation.

$$\alpha_1 \frac{k+\omega}{k_1(\omega)+\omega} + \alpha_2 \frac{k+\omega}{k_2(\omega)+\omega} = 1.$$

We define left hand side as a function $\psi(\omega, k)$ of ω and k. Thus we have

$$\psi_\omega = \frac{\partial \psi}{\partial \omega}(\omega, k) = \sum_{i=1}^{2} \alpha_i \frac{k_i(\omega) + \omega - (k+\omega)(k_i'(\omega)+1)}{(k_i(\omega)+\omega)^2}.$$

A sufficient condition for ω^* to be unique is

$$\psi_\omega(\omega, k) < 0 \text{ for } \omega = \omega^*. \tag{13}$$

Under (13), we can write equilibrium $\omega(k) = \omega^*$ as a function of k by implicit function theorem. Note that the function $\omega(k)$ is differentiable at the same time. And thus, the equality $\psi(\omega(k), k) = 1$ is an identity with respect to k. Therefore, we have

$$\psi_\omega \omega'(k) + \psi_k = 0, \ \psi_k \left(\stackrel{\text{def}}{=} \frac{\partial \psi}{\partial k} \right) = \sum_{i=1}^{2} \frac{\alpha_i}{k_i(\omega) + \omega} = \frac{1}{k + \omega} > 0.$$

Finally, we get to a starting point for comparative statics:

$$\frac{\mathrm{d}\omega}{\mathrm{d}k} (= \omega'(k)) = -\frac{\psi_k}{\psi_\omega} > 0, \quad \frac{\mathrm{d}k_i}{\mathrm{d}k} = \frac{\omega'}{\phi_i'} > 0, \ i = 1, 2.$$

3 Aggregation

In this section we are to define the aggregation of a Uzawa-economy to a Solow-economy and then to obtain possible candidates of a price of aggregate commodity and an aggregate production function.

Let a triplet $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ be a Uzawa-economy. Let us consider an economy with one commodity and two factors, i.e., capital and labor. Let $((K, L), F)$ be a one-sector economy, where (K, L) is identical with the factor endowments in the Uzawa-economy and where $F(\tilde{K}, \tilde{L})$ is a linearly homogenous, concave, and differentiable production function. The pair $((K, L), F)$ defines a Solow-economy. Let us define the temporary equilibrium of the Solow-economy.

Definition 2 (Solow's temporary Equilibrium) *Let $((K, L), F)$ be a Solow-economy. A pair of price vector and allocation $((\hat{\omega}, \hat{p}),\ (Y, K, L))$ is a Solow temporary equilibrium if and only if the pair satisfies following conditions* (S1) *and* (S2)*.*

(S1) (Y, K, L) *is a solution to the problem:*

$$\max_{\tilde{Y}, \tilde{K}, \tilde{L}} \hat{p}\tilde{Y} - \tilde{K} - \hat{\omega}\tilde{L} \ \ \textit{subject to} \ \ \tilde{Y} = F(\tilde{K}, \tilde{L}).$$

(S2) *The demand and supply for commodity are in balance. That is,*

$$K + \hat{\omega}L = \hat{p}F(K, L) = \hat{p}Y.$$

Solow (1956) assumes the saving ratio s is constant, then $s(K + \hat{\omega}L)$ is the demand for investment and $(1-s)(K+\hat{\omega}L)$ is the demand for consumption. And thus, (S2) is the equality between demand and supply for commodity. The condition (S2) holds whenever (S1) holds, by Euler's theorem. Constants s and $1-s$ in the Solow model correspond to α_1 and α_2 of the Uzawa-economy.

The necessary condition for (S1) is

$$\hat{p}f'(k) = 1,\ \hat{\omega} = \frac{f(k)}{f'(k)} - k,\ \text{where } f(k) = F(k,1),\ k = \frac{K}{L}.$$

Since F is concave, the condition is a sufficient condition for the maximization.

3.1 Definition of Aggregation

Let $((K,L),(F_i)_{i=1}^2,(\alpha_i)_{i=1}^2)$ be a Uzawa-economy. Let $(\omega,(p_i(\omega))_{i=1}^2)$ for $\omega \in \mathbb{R}_{++}$ be the price vector. Let $K_i(\omega), L_i(\omega),\ i = 1,2$ be factor demand functions in the Uzawa-economy. Total demands for factors are denoted by $K(\omega)$ and $L(\omega)$.

Definition 3 (Aggregate commodity prices and production) *Two functions* $p(\omega)$ *and* $F(\tilde{K},\tilde{L})$ *are said to be a* **price of aggregated commodity** *and an* **aggregate production function** *respectively if and only if these functions satisfy conditions* (a) *and* (b) *below.*

- **(a)** [**Aggregative Producer**] $F(\tilde{K},\tilde{L})$ *is differentiable, linearly homogenous, and concave. For an arbitrarily given* ω, $(K(\omega),L(\omega))$ *is a solution to the maximization problem*:

$$\max_{\tilde{K},\tilde{L}} p(\omega)F(\tilde{K},\tilde{L}) - \tilde{K} - \omega\tilde{L}.$$

- **(b)** [**Conservation of product value**]. *The following equality is an identity with respect to* ω.

$$p(\omega)F(K(\omega),L(\omega)) = p_1(\omega)F_1(K_1(\omega),L_1(\omega)) + p_2(\omega)F_2(K_2(\omega),L_2(\omega)).$$

We call a pair of functions $(p(\omega),F(\tilde{K},\tilde{L}))$ an **aggregate pair**. Note that conditions (a) and (b) in Definition 3 describe desirable properties that we claim to the aggregate pair if such exist.

Definition 4 (Aggregation of economy) The Uzawa-economy $((K, L), (F_i)_{i=1}^2,$ $(\alpha_i)_{i=1}^2)$ **is said to be aggregated into a Solow-economy** $((K, L), F)$ *if and only if* $(p(\omega),\ F(\tilde{K}, \tilde{L}))$ *is the aggregate pair satisfying conditions* (a) *and* (b) *in Definition* 3 *and besides the Uzawa temporary equilibrium* $((\omega^*, (p_i(\omega^*))_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ *and a pair* $((\omega^*, p(\omega^*)), (Y, K, L))$ *satisfy conditions* (c) *and* (d).

(**c**) *Following equalities hold.*

$$K(\omega^*) = K,\ \ L(\omega^*) = L,\ \ Y = F(K, L).$$

(**d**) *The pair of price and aggregate allocation* $((\omega^*, p(\omega^*)), (Y, K, L))$ *is a Solow temporary equilibrium in the Solow-economy* $((K, L), F)$, *where* $Y = F(K, L)$.

The condition (d) in Definition 4 may be repetitive. We, however, dare to demonstrate (d) since it is an important condition. The condition (c) together with (b) implies that

$$p(\omega^*)Y = p_1(\omega^*)Y_1^* + p_2(\omega^*)Y_2^*.$$

Assume that there exists an aggregate pair $(p(\omega), F)$ satisfying conditions (a) and (b) in Definition 3. Since F is differentiable, linearly homogenous, and concave and since $(K(\omega), L(\omega))$ is an inner solution to the problem in (a), the maximization in (a) is equivalent to the following equations.

$$p(\omega)\frac{\partial F}{\partial K}(K(\omega), L(\omega)) = 1,\quad p(\omega)\frac{\partial F}{\partial L}(K(\omega), L(\omega)) = \omega.$$

Define $f(\tilde{k}) = F(\tilde{k}, 1),\ k(\omega) = K(\omega)/L(\omega)$ where $\tilde{k} = \tilde{K}/\tilde{L}$. Then the above equations are equivalent to

$$p(\omega)f'(k(\omega)) = 1,\quad \frac{f'(k(\omega))}{f(k(\omega))} = \frac{1}{k(\omega) + \omega}.$$

And thus, we obtain an equality corresponding to (7).

3.2 Candidate of Aggregate Pair satisfying (a) and (b)

In this subsection, we are to find a possible aggregate pair. That is, we assume that there exist p and F satisfying conditions (a) and (b) to search possible forms of p and F. A candidate of (p, F) is obtained in the following theorem.

Theorem 1 *Let* $((K,L),(F_i)_{i=1}^2,(\alpha_i)_{i=1}^2)$ *be a Uzawa-economy. Let* $((\omega^*,(p_i(\omega^*))_{i=1}^2),(Y_i^*,K_i^*,L_i^*)_{i=1}^2)$ *be the Uzawa temporary equilibrium. Suppose that the Uzawa temporary equilibrium is unique. If there exists an aggregate pair* $(p(\omega),F(\tilde{K},\tilde{L}))$ *satisfying conditions* (a) *and* (b) *in Definition* 3*, then following two equalities hold*:

$$p(\omega^*) = (p_1(\omega^*))^{\alpha_1}(p_2(\omega^*))^{\alpha_2}, \tag{14}$$

$$F(K,L) = \left(\frac{F_1(K_1^*,L_1^*)}{\alpha_1}\right)^{\alpha_1}\left(\frac{F_2(K_2^*,L_2^*)}{\alpha_2}\right)^{\alpha_2}. \tag{15}$$

The proof of Theorem 1 is given in Appendix. Note that $K_i^*,L_i^*, i=1,2$ in the right hand side of (15) depend on K,L and thus (15) is represented as a function $F(K,L)$. It is easy for us to prove that (14) holds when we have the aggregate production function F as indicated in (15). The equation (12) together with (15) leads us to

$$p_1(\omega^*)^{\alpha_1}p_2(\omega^*)^{\alpha_2}F(K,L) = p_1(\omega^*)F_1(K_1^*,L_1^*) + p_2(\omega^*)F_2(K_2^*,L_2^*).$$

This equality and the condition (b) in Definition 3 lead us to (14).

Note that we obtain equations (14) and (15) as a necessary condition for the aggregate pair satisfying conditions (a) and (b). On the other hand, we have not shown that the condition (a) hold for any ω and that the function (15) is concave and linearly homogenous. In the following Section, we shall show that (15) is the aggregate production function.

Cobb Douglas example Doi, Fujii, Horie, Iritani, Sato and Yasuoka(2021) gives an example of (15). Suppose that each sector i has a production function of

$$F_i(K_i,L_i) = A_i{K_i}^{\theta_i}{L_i}^{1-\theta_i},$$

where A_i and θ_i denote sector i's TFP and the capital share rate $(0<\theta_i<1)$ respectively, $i=1,2$. Define θ and A as follows:

$$\theta = \alpha_1\theta_1 + \alpha_2\theta_2,$$

$$A = \prod_{i=1}^{2}\left(\frac{A_i{\theta_i}^{\theta_i}(1-\theta_i)^{1-\theta_i}}{\theta^\theta(1-\theta)^{1-\theta}}\right)^{\alpha_i}.$$

The formula (15) in this example is

$$F(K,L) = AK^\theta L^{1-\theta}.$$

In this case, the candidate of aggregate production function obtained here is linearly homogenous, concave, and differentiable.

4 Aggregate Pair

We have found a possible candidate of aggregate production function in Theorem 1. In this section, we present functions p and F and establish properties of F that are required for the aggregate production function. In order to attain the goal, we give a new presentation for the temporary equilibrium of Uzawa-economy.

4.1 Temporary Equilibrium – a New Presentation –

Our main objectives of this subsection is to achieve two theorems. One is the equivalence theorem (Theorem 2, below). By this theorem, we can give another full-length portrait of the Uzawa temporary equilibrium by a simple problem (16) below. The other is the uniqueness theorem (Theorem 3, below) which shows that the Uzawa temporary equilibrium is unique.

Let $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ be a Uzawa economy. Let us define an artificial maximization problem such as follows:

$$\max_{y_i, K_i, L_i} \left(\frac{y_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2}{\alpha_2}\right)^{\alpha_2} \text{ subject to } \begin{cases} K_1 + K_2 \leqq K,\ L_1 + L_2 \leqq L, \\ y_i \leqq F_i(K_i, L_i),\ i = 1, 2. \end{cases} \tag{16}$$

It is Baqaee and Farhi (2019) who introduced this type of problem to obtain an aggregate production function. Their key concept is their objective function D_0, the aggregator function, variables of which are net products. Maximizing D_0 under resource constraints gives each endogenous variable a functional form of exogenous parameters. Finally, D_0 can be regarded as the function of parameters, e.g., endowments of primal factors. This is their aggregate production function. Baqaee and Farhi (2019) remains unanswered a concrete form of D_0 and do not try to construct an aggregate economy. Now, we have a possible form of aggregate production function as in (15) in Theorem 1. Then we employ this type of objective function $(y_1/\alpha_1)^{\alpha_1} (y_2/\alpha_2)^{\alpha_2}$ for that in (16).

Let A be a set of all $(y_i, K_i, L_i)_{i=1}^2$ satisfying constraints in (16). The set A represents the attainable set of the Uzawa-economy and is convex and compact. Sets $A(K, L)$ and $A(y)$ denote projections from A to $(K_i, L_i)_{i=1}^2$-plane and (y_1, y_2)-plane respectively. $A(y)$ is so called a downside area of the transformation curve. The set $A(y)$ is also a convex compact set. We can observe that an upper contour set of the objective function in (16) is a strictly convex set in $\mathbb{R}^2_{++}$. Therefore next Lemma is obvious.

Lemma 2 *A solution to the problem* (16) *exists and is an inner solution.*

The Lagrangian of the problem (16) is

$$\mathcal{L} = \left(\frac{y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{y_2}{\alpha_2}\right)^{\alpha_2} + \sum_{i=1}^{2}\lambda_i(F_i(K_i, L_i) - y_i)$$
$$+ \mu(K - K_1 - K_2) + \delta(L - L_1 - L_2).$$

The Kuhn-Tucker condition associating with (16) is a following set of equations:

$$\alpha_i \frac{(y_1/\alpha_1)^{\alpha_1}(y_2/\alpha_2)^{\alpha_2}}{y_i} - \lambda_i = 0,\ i = 1, 2 \tag{17}$$

$$\lambda_i \frac{\partial F_i}{\partial K_i} - \mu = 0, \qquad i = 1, 2 \tag{18}$$

$$\lambda_i \frac{\partial F_i}{\partial L_i} - \delta = 0, \qquad i = 1, 2 \tag{19}$$

$$F_i(K_i, L_i) - y_i = 0, \qquad i = 1, 2 \tag{20}$$

$$K - K_1 - K_2 = 0,\ L - L_1 - L_2 = 0. \tag{21}$$

Let $(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2$ be the solution to (16). Substituting (y_1^{**}, y_2^{**}) to (y_1, y_2) in (17), we have values of $\lambda_i, i = 1, 2$, and sequentially, we have μ from (18) and δ from (19). Clearly, the allocation $(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2$ satisfies (20) and (21).

Theorem 2 (Equivalence Theorem) *The Uzawa temporary equilibrium is equivalent to the solution to* (16). *That is to say, following conditions* (i) *and* (ii) *hold.*

(i) *Let* $((\omega^*, (p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ *be the Uzawa temporary equilibrium. Define*

$$p^* = p_1^{*\alpha_1} p_2^{*\alpha_2} \tag{22}$$

$$\lambda_i = \frac{p_i^*}{p^*}, i = 1, 2,\ \mu = \frac{1}{p^*},\ \delta = \frac{\omega^*}{p^*}. \tag{23}$$

Then the pair $((\lambda_1, \lambda_2, \mu, \delta), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ *is a solution to the system* (17), (18), (19), (20) *and* (21). *Furthermore, the allocation* $(Y_i^*, K_i^*, L_i^*)_{i=1}^2$ *is a solution to the problem* (16).

(ii) *Let* $((\lambda_1, \lambda_2, \mu, \delta), (y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2)$ *be a solution of Kuhn-Tucker condition associating with* (16). *Define*

$$\omega^* = \frac{\delta}{\mu},\ p_i^* = \frac{\lambda_i}{\mu}, i = 1, 2, \tag{24}$$

and the pair $((\omega^*, (p_i^*)_{i=1}^2), (y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2)$ *is the Uzawa temporary equilibrium.*

Theorem 2 is proven in Appendix.

This theorem establishes the equivalence between the Uzawa temporary equilibrium and the solution to (16). It is noteworthy that p^* in (22) is identical with (14). The theorem 2 is the new knowledge in the theory of two-sector growth model (see, e.g., Burmeister and Dobell (1970)).

Our next objective is to establish the uniqueness of the solution to (16).

Theorem 3 *Suppose that a Uzawa-economy* $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ *is given. The solution to the problem* (16) *is unique, therefore, the Uzawa temporary equilibrium is also unique.*

The proof is relegated to Appendix.

Let a pair $((\omega^*, (p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be the unique temporary equilibrium of a Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. And let $(\omega, (p_i(\omega))_{i=1}^2)$ be a price vector for $\omega \in \mathbb{R}_{++}$ satisfying $p_i^* = p_i(\omega^*), i = 1, 2$. And thus, we can define functions $p(\omega)$ and $F(K, L)$ as :

$$p(\omega) = (p_1(\omega))^{\alpha_1} (p_2(\omega))^{\alpha_2}, \tag{25}$$

$$F(K, L) = \left(\frac{F_1(K_1^*, L_1^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2^*, L_2^*)}{\alpha_2}\right)^{\alpha_2}. \tag{26}$$

By this definition and (12) we have a local version of the condition (b) hold, i.e.,

$$p(\omega^*)F(K, L) = p_1(\omega^*)F_1(K_1^*, L_1^*) + p_2(\omega^*)F_2(K_2^*, L_2^*). \tag{27}$$

In following sections, we shall show that (25) and (26) constitute the aggregate pair satisfying conditions (a) and (b).

4.2 Differentiability

We have now two ways of finding the Uzawa temporary equilibrium. One is presented in Section 2.3 and the other in Section 4.1. The dependency relationships of variables are different in each way.

In the way presented in Section 2.3, dependency relationships of variables is depicted in the following format:

$$(K, L) \xrightarrow{\text{labor market}} \omega \xrightarrow{(4)} (k_i, p_i)_{i=1}^2 \xrightarrow{(9),\ (10)} (\rho_i, K_i, L_i)_{i=1}^2 \xrightarrow{\mathrm{F_i}} (Y_i)_{i=1}^2.$$

On the other hand, the way developed in Section 4.1 is very simple. Every variable is obtained as a function of (K, L) by the maximization problem (16). To distinguish the dependency relationships of variables determined in (16) from those in Section 2.3, we write functions by using brackets as follows:

$$K_i[K,L], L_i[K,L], Y_i[K,L],\ i=1,2.$$

By Theorem 3, we have identities:

$$\begin{cases} k_i(\omega(K/L))\rho_i(\omega(K/L))L = K_i[K,L], & i=1,2 \\ \rho_i(\omega(K/L))L = L_i[K,L], & i=1,2 \\ Y_i(K_i(\omega(K/L)), L_i(\omega(K/L))) = F_i(K_i[K,L], L_i[K,L]) = Y_i[K,L], & i=1,2. \end{cases} \tag{28}$$

Functions in the left hand side of (28) are those obtained in Section 2.3, and functions in the right hand side those in (16). And thus, we have an identity such that

$$F(K,L) = \prod_{i=1}^{2} \left\{ \frac{f_i(k_i(\omega(K/L)))L_i(\omega(K/L))}{\alpha_i} \right\}^{\alpha_i} = \prod_{i=1}^{2} \left(\frac{F_i(K_i[K,L], L_i[K,L])}{\alpha_i} \right)^{\alpha_i}. \tag{29}$$

Differentiability of "functions with brackets" are assured by the condition that the Jacobian determinant of the systems of Kuhn Tucker condition does not vanish. On the other hand, differentiability of "functions with parentheses" is assured by assuming (13) since differentiating these variables is an operation which is comparative statics intrinsically. We can differentiate "functions with brackets" under assumption (13) because equalities in (28) are identities.

Therefore, next Lemma is obvious.

Lemma 3 $K_i[K,L],\ L_i[K,L]$, *and* $Y_i[K,L],\ i=1,2$ *are differentiable with respect to* (K,L) *if* (13) *is true for* $(K,L) \in \mathbb{R}^2_{++}$.

We can make a conjecture that assuming (13) is equivalent to assuming that the Jacobian determinant of the systems of Kuhn Tucker condition does not vanish. We, however, avoid getting involved with this problem.

4.3 Linear homogeneity, monotonicity, and concavity of F

Let $((\omega^*, (p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be the temporary equilibrium of a Uzawa-economy $((K,L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. Let t be an arbitrary positive real. Then a pair of the price vector and the allocation $((\omega^*, (p_i^*)_{i=1}^2), (tY_i^*, tK_i^*, tL_i^*)_{i=1}^2)$ is a temporary equilibrium of a Uzawa-economy $((tK, tL), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. Thus we acquire

$$F(tK, tL) = \left(\frac{F_1(tK_1^*, tL_1^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(tK_2^*, tL_2^*)}{\alpha_2}\right)^{\alpha_2} = tF(K, L).$$

We are to show that F is monotonously increasing. Let $X = (K, L)$ and $X' = (K', L')$ be two pairs of positive factor endowments satisfying $X \leqq X'$ and $X \neq X'$. Let $(Y_1[X], Y_2[X])$ be a vector of two products in the temporary equilibrium associating with the Uzawa-economy $(X, (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. In the same way, we have $(Y_1[X'], Y_2[X'])$ for the Uzawa-economy $(X', (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. Then $A(X) \subsetneq A(X')$ since $X \leqq X'$ and $X \neq X'$. This implies by (16) that

$$F(X) < F(X').$$

And thus, F is strictly increasing with respect to $(K, L) \in \mathbb{R}_{++}^2$.

We can summarize these as follows:

Theorem 4 *The function F defined by* (26) *is monotonous and homogenous of degree one.*

We show the concavity of the function F in the next theorem.

Theorem 5 *The function F is concave with respect to $X = (K, L) \in \mathbb{R}_{++}^2$.*

The proof of this theorem is relegated to Appendix. A concave function is differentiable except points of measure zero of the domain.[2] This implies that assuming (13) is fairly weak.

5 Aggregation Possibility Theorems

Let $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ be a Uzawa-economy. Let ω be a positive real. By the discussions made previously, we have functions $k_i(\omega)$, $p_i(\omega) = 1/f_i'(k_i(\omega))$, $Y_i(\omega)$, $L_i(\omega)$, $K_i(\omega)$, $i = 1, 2$, and total factor demand functions for factors $L(\omega), K(\omega)$.

Now, we introduce a concept of **trivial equilibrium**.

[2] See Theorem 25.5 in Rockafellar(1970, page 246).

Theorem 6 *Let ω be an arbitrary positive wage-rental ratio. Let $(\omega, (p_i(\omega))_{i=1}^2)$ and $(Y_i(\omega), K_i(\omega), L_i(\omega))_{i=1}^2$ be a pair of price vector and production vectors in the Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. Define $(\bar{K}, \bar{L}) = (K_1(\omega) + K_2(\omega), L_1(\omega) + L_2(\omega))$. Then the pair*

$$\big((\omega, (p_i(\omega))_{i=1}^2), (Y_i(\omega), K_i(\omega), L_i(\omega))_{i=1}^2\big)$$

is a temporary equilibrium in a new Uzawa-economy $((\bar{K}, \bar{L}), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. This temporary equilibrium is said to be a "trivial equilibrium". A trivial equilibrium is a solution to the problem (16) *when $(K, L) = (\bar{K}, \bar{L})$.*

The proof is relegated to Appendix.

Theorem 6 is not trivial as it seems but is very important. Let us show its powerfulness. The equation (12) and the definition of F imply that the condition (b) holds locally at the temporary equilibrium of the Uzawa-economy. Here, Theorem 6 states that $((\omega, (p_i(\omega))_{i=1}^2), (Y_i(\omega), K_i(\omega),\ L_i(\omega))_{i=1}^2)$ is also a temporary equilibrium of a Uzawa-economy $((K(\omega), L(\omega)), (F_i)_{i=1}^2,\ (\alpha_i)_{i=1}^2)$.

And by (12) and (25), we have

$$p(\omega)F(K(\omega), L(\omega)) = p_1(\omega)F_1(K_1(\omega), L_1(\omega)) + p_2(\omega)F_2(K_2(\omega), L_2(\omega)). \tag{30}$$

This equality holds for any positive ω. Then, the condition (b) in Definition 3 holds.

Now, we assume following Assumption 2.[3]

Assumption 2 *The solution to the problem* (16) *is differentiable with respect to the parameter $(K, L) \in \mathbb{R}^2_{++}$.*

Lemma 4 *Under Assumption* 2*, the condition* (a) *in Definition* 3 *holds locally. In other words, let $((\omega^*, (p_i(\omega^*))_{i=1}^2),\ (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be the temporary equilibrium of a Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. Then, following property* (aL) *is true.*

(aL) *Define $p(\omega^*) = (p_1(\omega^*))^{\alpha_1}(p_2(\omega^*))^{\alpha_2}$. Then (K, L) is a solution to the problem:*

$$\max_{\tilde{K}, \tilde{L}} p(\omega^*)F(\tilde{K}, \tilde{L}) - \tilde{K} - \omega^*\tilde{L}.$$

Proof of Lemma 4 is given in Appendix.

We are fully equipped to show that the condition (a) in Definition 3 holds. Let ω be a positive real. Define $(\bar{K}, \bar{L}) = (K(\omega), L(\omega))$. We can construct a Uzawa-economy

[3] We can dispense with this assumption if the Jacobian of the simultaneous equations (17), (18), (19), (20), (21) does not vanish or if we assume (13), which is the sufficient condition in Lemma 3.

$((\bar{K}, \bar{L}), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. By Theorem 6, the pair $((\omega, (p_i(\omega))_{i=1}^2), (Y_i(\omega), K_i(\omega), L_i(\omega))_{i=1}^2)$ is the trivial equilibrium in the new Uzawa-economy. And thus, we can apply Lemma 4 to the trivial equilibrium. This implies the condition (a) in Definition 3 holds if $p(\omega)$ is determined by (25).

We have already established the condition (b) in Definition 3 when $p(\omega)$ is determined by (25). By these, we have the following theorem.

Theorem 7 (Aggregation Possibility Theorem I) *Define a function F and a price of aggregated commodity $p(\omega)$ by* (26) *and* (25) *respectively. Under Assumption* 2*, the functions $p(\omega)$ and F satisfy conditions* (a) *and* (b) *in Definition* 3*. That is, $p(\omega)$ is the price of aggregated commodity. And F is an aggregate production function.*

Furthermore, we have by (30):

$$p(\omega)F(K(\omega), L(\omega)) = K(\omega) + \omega L(\omega). \tag{31}$$

Of course, the equality (31) holds when $\omega = \omega^*$. Therefore, the condition (c) in Definition 4 holds. Furthermore, profit maximization of aggregate production under prices $(\omega^*, p(\omega^*))$ is attained at (K, L) by Lemma 4. This implies that $((\omega^*, p(\omega^*)), (Y^*, K, L))$ is a Solow equilibrium. This is nothing but the condition (d) in Definition 4.

These results are summarized up in the following theorem.

Theorem 8 (Aggregation Possibility Theorem II) *Define a function F and a price of aggregated commodity $p(\omega)$ by* (26) *and* (25) *respectively. Under Assumption* 2*, the Uzawa-economy $((K, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ is aggregated into the Solow-economy $((K, L), F)$.*

A set of Theorems 7 and 8 is an affirmative answer to the aggregation of the Uzawa-economy into the Solow-economy.

6 Concluding Remarks

In this paper, we have shown that the temporary equilibrium of a Uzawa economy can be aggregated into that of Solow economy successfully.

Our scope is, however, restrictive in following two respects. One is whether the aggregation of the current two-sector model into the one-sector model can be extended to the aggregation from the n-sector model to the one-sector model. This point can be easily resolved by extending this paper. Let us consider an n-sector economy where the i-th sector has a neoclassical production function $F_i(K_i, L_i),\ i = 1, 2, \ldots, n$. Let positive constant α_i be a coefficient of expenditure to i-th sector, $i = 1, 2, \ldots, n$ satisfying $\sum_{i=1}^{n} \alpha_i = 1$. Suppose that a temporary equilibrium of n-sector model $((\omega^*,\ (p_i^*)_{i=1}^n), (Y_i^*, K_i^*, L_i^*)_{i=1}^n)$ exists. Then we can show that a pair

$$p = \prod_{i=1}^{n} (p_i^*)^{\alpha_i}, \quad F(K, L) = \prod_{i=1}^{n} \left(\frac{F_i(K_i^*, L_i^*)}{\alpha_i} \right)^{\alpha_i}$$

is an aggregate pair of aggregated economy.

Another point is whether the aggregation procedure obtained in the present paper can be applies to dynamic paths. The two-sector model of this paper, which focuses on temporary equilibrium, does not describe the accumulation of capital. As in the usual Uzawa two-sector model, the first sector produces the investment goods and the second sector the consumption goods. Then the supply and demand balance of products of first sector describes the dynamic path. In this case, the expenditure coefficient α_1 is nothing but the savings rate s. An important issue that arises at this time is the following. Let Y be a temporal national income measured by wages which is identical in both economies. The amount of saving is sY which is also identical in two economies. Therefore, we ca see that

$$\text{investment in two sector economy} = \dot{K} = \frac{sY}{p_1}$$
$$\text{investment in aggregate economy} = \dot{K} = \frac{sY}{{p_1}^s {p_2}^{1-s}}.$$

This means that the dynamic path may be different in two economies, that is, that we need develop the more sophisticated procedure for aggregation of economy which is consistent not only temporarily but also dynamically.

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7 Mathematical Appendix

A.1 Proof of Lemma 1

Assume that F is a function satisfying Assumption 1. Let P be a set $\{(y, K, L) \in \mathbb{R}^3_+ \,|\, y \leqq F(K, L)\}$. P is a closed convex cone, since F_i is continuous, linearly homogenous, and concave. Let ∂P be a set of the boundary points in P production level of which is positive. Suppose that there exists a line segment on ∂P such that its extended line <u>does not</u> pass through the origin. Let $\bar{\xi} = (\bar{y}, \bar{K}, \bar{L}), \hat{\xi} = (\hat{y}, \hat{K}, \hat{L}) \in \mathbb{R}^3_{++}$ be two distinct points on the segment. Since the line $\{\mu(\hat{y}, \hat{K}, \hat{L}) \,|\, \mu > 0\}$ is on ∂P. There exists $\mu > 0$ such that $\mu\hat{y} = \bar{y}$. We can assume without loss of generality that $\mu = 1$. It is also obvious that $(\hat{K}, \hat{L}) \neq (\bar{K}, \bar{L})$ and that the segment is on ∂P. Define $(K(\lambda), L(\lambda)) = \lambda(\bar{K}, \bar{L}) + (1 - \lambda)(\hat{K}, \hat{L})$ for $0 \leqq \lambda \leqq 1$. Clearly, it holds that $\bar{y} = F(K(\lambda), L(\lambda))$ and that the point $(\bar{y}, K(\lambda), L(\lambda))$ is on ∂P. This implies that $\{(K(\lambda), L(\lambda)) \,|\, 0 \leqq \lambda \leqq 1\}$ is on an isoquant associating with $\bar{y}$. This, however, is impossible since the the second derivative along the isoquant is negative, i.e., the marginal rate of substitution is strictly decreasing by A3 in Assumption 1. This implies that there are no line segments on ∂P such that its extended line does not pass through the origin. ∎

A.2 Proof of Theorem 1

By $(3')$ and the definitions of $L_i(\omega)$ and $K_i(\omega)$ in (9), we can rewrite the condition (b) in Definition 3 as follows:

$$\begin{aligned} p(\omega)F(K(\omega), L(\omega)) &= p_1(\omega)F_1(K_1(\omega), L_1(\omega)) + p_2(\omega)F_2(K_2(\omega), L_2(\omega)) \\ &= K + \omega L. \end{aligned}$$

Since F is linearly homogenous, the left hand side is identical with $p(\omega)f(k(\omega))L(\omega)$. Therefore the condition (b) is expressed by

$$p(\omega)f(k(\omega)) = (k + \omega)\frac{L}{L(\omega)} = (k + \omega)\frac{1}{\sum_{i=1}^{2} \alpha_i \frac{k+\omega}{k_i(\omega)+\omega}}.$$

Furthermore the condition (a) leads us to the conclusion that $(K(\omega), L(\omega))$ is an inner solution to the profit maximizing problem. Since $p(\omega) = 1/f'(k(\omega))$, the condition (b) is rearranged further to

$$\begin{aligned} \frac{f'(k(\omega))}{f(k(\omega))} &= \sum_{i=1}^{2} \frac{\alpha_i}{k_i(\omega) + \omega} = \alpha_1 \frac{f_1'(k_1(\omega))}{f_1(k_1(\omega))} + \alpha_2 \frac{f_2'(k_2(\omega))}{f_1(k_2(\omega))} \\ \frac{1}{k(\omega) + \omega} &= \frac{\alpha_1}{k_1(\omega) + \omega} + \frac{\alpha_2}{k_2(\omega) + \omega}. \end{aligned} \quad (32)$$

Applying (7) to each term in (32), we have

$$\log(k(\omega)+\omega) - \log f(k(\omega)) + C = \sum_{i=1}^{2} \alpha_i \left(\log(\omega + k_i(\omega)) - \log f_i(k_i(\omega)) + C_i\right)$$

where $C, C_i, i = 1, 2$ are integral constants. And thus we know:

$$\log f(k(\omega)) = \log\left(f_1(k_1(\omega))^{\alpha_1} f_2(k_2(\omega)^{\alpha_2}\right) + \log \frac{\omega + k(\omega)}{(\omega + k_1(\omega))^{\alpha_1}(\omega + k_2(\omega))^{\alpha_2}} + C.$$

Let the integral constant C be zero, which means a suitable choice of unit of K or L. Here we focus our attention on equilibria. It holds that $k(\omega^*) = k$ in equilibrium. Making use of (11), we can rewrite the second term in the right hand side as follows.

$$\begin{aligned}
\text{the second term in RHS }|_{\omega=\omega^*} &= \log \left(\frac{\omega^* + k(\omega^*)}{\omega^* + k_1(\omega^*)}\right)^{\alpha_1} \left(\frac{\omega^* + k(\omega^*)}{\omega^* + k_2(\omega^*)}\right)^{\alpha_2} \\
&= \log \left(\frac{\omega^* + k}{\omega^* + k_1(\omega^*)}\right)^{\alpha_1} \left(\frac{\omega^* + k}{\omega^* + k_2(\omega^*)}\right)^{\alpha_2} \\
&= \log \left(\frac{L_1(\omega^*)}{\alpha_1 L}\right)^{\alpha_1} \left(\frac{L_2(\omega^*)}{\alpha_2 L}\right)^{\alpha_2} \\
&= \log \frac{(L_1(\omega^*)/\alpha_1)^{\alpha_1} (L_2(\omega^*)/\alpha_2)^{\alpha_2}}{L}.
\end{aligned}$$

Substitute this relation to the original equality and we have for $\omega = \omega^*$

$$\log f(k)L = \log\left[\left(\frac{f_1(k_1(\omega^*))L_1(\omega^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{f_2(k_2(\omega^*))L_2(\omega^*)}{\alpha_2}\right)^{\alpha_2}\right].$$

Finally, this implies

$$\begin{aligned}
f(k) &= \left(\frac{f_1(k_1(\omega^*))\rho_1(\omega^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{f_2(k_2(\omega^*))\rho_2(\omega^*)}{\alpha_2}\right)^{\alpha_2}, \\
F(K, L) &= \left(\frac{F_1(K_1^*, L_1^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2^*, L_2^*)}{\alpha_2}\right)^{\alpha_2}.
\end{aligned}$$

Note that $F(K, L) = f(K/L)L$ in the second equation. And thus, we obtain the desired conclusion. ∎

A.3 Proof of Theorem 2

Proof of (i). Let $((\omega^*, (p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be a Uzawa temporary equilibrium. By (22) and (23), two equalities in Kuhn-Tucker condition (18), (19) hold:

$$\lambda_i \frac{\partial F_i}{\partial K_i}(K_i^*, L_i^*) - \mu = \frac{1}{p^*}\left(p_i^* \frac{\partial F_i}{\partial K_i}(K_i^*, L_i^*) - 1\right) = 0,$$
$$\lambda_i \frac{\partial F_i}{\partial L_i}(K_i^*, L_i^*) - \delta = \frac{1}{p^*}\left(p_i^* \frac{\partial F_i}{\partial L_i}(K_i^*, L_i^*) - \omega^*\right) = 0.$$

Equalities (20) and (21) hold obviously. The equilibrium of commodity i enables us to know:

$$\alpha_i \left(\frac{Y_1^*}{\alpha_1}\right)^{\alpha_1} \left(\frac{Y_2^*}{\alpha_2}\right)^{\alpha_2} - \lambda_i Y_i^* = \frac{1}{p^*}\left(\alpha_i \left(\frac{p_1^* Y_1^*}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* Y_2^*}{\alpha_2}\right)^{\alpha_2} - p_i^* Y_i^*\right)$$
$$= \frac{1}{p^*}\left(\alpha_i (K + \omega^* L) - p_i^* Y_i^*\right) = 0.$$

Therefore we have

$$\alpha_i \frac{\left(\frac{Y_1^*}{\alpha_1}\right)^{\alpha_1 - 1} \left(\frac{Y_2^*}{\alpha_2}\right)^{\alpha_2}}{Y_i^*} - \lambda_i = 0, \ i = 1, 2.$$

This implies that the first assertion in (i) holds. Finally, the allocation $(Y_i^*, K_i^*, L_i^*)_{i=1}^2$ satisfying the Kuhn-Tucker condition (17), (18), (19), (20), and (21) is a solution to the problem (16) by Magasarian (1969, Theorem 7.2.1). This is the second assertion in (i).

Proof of (ii). Let $((\lambda_1, \lambda_2, \mu, \delta), (y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2)$ be a solution to Kuhn-Tucker condition associating to (16). Note that these variables are positive. We define ω^*, $p_i^*, i = 1, 2$ by (24). Since $p_i^* \times \mu = \lambda_i, i = 1, 2$, the relations (18), (19) are respectively rearranged to

$$p_i^* \frac{\partial F_i}{\partial K_i}(K_i^{**}, L_i^{**}) = 1, \quad p_i^* \frac{\partial F_i}{\partial L_i}(K_i^{**}, L_i^{**}) = \omega^*.$$

This implies that (K_i^{**}, L_i^{**}) is a profit maximizer of the producer i when prices are (ω^*, p_i^*). Factor markets are in balance because (21) holds. Let us consider the commodity markets. By (17), we have

$$\left(\frac{y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2^{**}}{\alpha_2}\right)^{\alpha_2} = \lambda_i \frac{y_i^{**}}{\alpha_i}, \ i = 1, 2.$$

This equality implies $1 = \prod_{i=1}^{2} \lambda_i^{\alpha_i}$. By the definitions of ω^* and $p_i^*, i = 1, 2$ in (24) we can obtain $\mu^{-1} = \prod_{i=1}^{2} (p_i^*)^{\alpha_i}$. Furthermore, by (20) we have:

$$y_i^{**} = F_i(K_i^{**}, L_i^{**}) = \frac{\partial F_i}{\partial K_i}(K_i^{**}, L_i^{**})K_i^{**} + \frac{\partial F_i}{\partial L_i}(K_i^{**}, L_i^{**})L_i^{**}.$$

By (24), we have known that $p_i^* = \lambda_i/\mu$ and $\mu^{-1} = \prod_{i=1}^{2} (p_i^*)^{\alpha_i}$. Multiplying both sides of (17) by $\prod_{i=1}^{2} (p_i^*)^{\alpha_i}$ $(= \mu^{-1})$, we have

$$\alpha_i \left(\frac{p_1^* y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* y_2^{**}}{\alpha_2}\right)^{\alpha_2} = p_i^* y_i^{**},\ \ i = 1, 2. \tag{33}$$

Add up above equalities with respect to i and we obtain

$$\left(\frac{p_1^* y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* y_2^{**}}{\alpha_2}\right)^{\alpha_2} = p_1^* y_1^{**} + p_2^* y_2^{**} = K + \omega^* L.$$

We can substitute this into the same term in (33). And taking (20) into account, we finally arrive at a conclusion:

$$\alpha_i \left(K + \omega^* L\right) = p_i^* y_i^{**} = p_i^* F_i(K_i^{**} L_i^{**}),\ \ i = 1, 2.$$

This implies that two commodity markets are in balance. And thus, a pair of prices and allocation $\left((\omega^*, (p_i^*)_{i=1}^{2}), (y_i^{**}, K_i^{**}, L_i^{**})\right)$ is the Uzawa temporary equilibrium. ∎

A.4 Proof of Theorem 3

Let $(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^{2}$ be a solution to the maximization problem (16). Note that (y_1^{**}, y_2^{**}) is in $A(y)$. Defining $g(y_1, y_2) = \left(\frac{y_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2}{\alpha_2}\right)^{\alpha_2}$, we simplify the maximization problem as follows:

$$\max g(y_1, y_2) \text{ subject to } (y_1, y_2) \in A(y).$$

The solution to this problem is unique and is identical with (y_1^{**}, y_2^{**}). Therefore, any solution $(y_i^*, K_i^*, L_i^*)_{i=1}^{2}$ to the problem (16) satisfies $(y_1^{**}, y_2^{**}) = (y_1^*, y_2^*)$. Let $(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^{2}$ and $(y_i^{**}, K_i^*, L_i^*)_{i=1}^{2}$ be two solutions to (16). Suppose that two factor allocations $(K_i^*, L_i^*)_{i=1}^{2}$ and $(K_i^{**}, L_i^{**})_{i=1}^{2}$ were in a relation that

$$\exists i\ \forall \mu_i \in \mathbb{R}_{++} \text{ such that } (K_i^*, L_i^*) \neq \mu_i(K_i^{**}, L_i^{**}). \tag{34}$$

Because of convexity of $A(K, L)$, it holds that

$$(K_i^{\#}, L_i^{\#})_{i=1}^{2} = \left(\frac{K_i^{**} + K_i^*}{2}, \frac{L_i^{**} + L_i^*}{2}\right)_{i=1}^{2} \in A(K, L).$$

Defining $y_i^{\#} = F_i(K_i^{\#}, L_i^{\#}), i = 1, 2$, we have

$$g(y_1^{\#}, y_2^{\#}) \geqq g(y_1^{**}, y_2^{**})$$

because g is a concave function. The fact that $(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2$ is a solution to the problem (16) leads us to another fact that $(y_i^{\#}, K_i^{\#}, L_i^{\#})_{i=1}^2$ is a solution to (16). In addition to this the equality $g(y_1^{\#}, y_2^{\#}) = g(y_1^{**}, y_2^{**})$ holds. This is a contradiction to Lemma 1. Therefore, the negation of (34) holds. That is,

$$\forall i \ \exists \mu_i \in \mathbb{R}_{++} \text{ and } (K_i^*, L_i^*) = \mu_i(K_i^{**}, L_i^{**}).$$

Then $\mu_i = 1$ holds since $y_i^{**} = F_i(K_i^*, L_i^*) = F_i(K_i^{**}, L_i^{**}), i = 1, 2$. This implies

$$(y_i^{**}, K_i^{**}, L_i^{**})_{i=1}^2 = (y_i^{**}, K_i^*, L_i^*)_{i=1}^2.$$

This establishes the uniqueness of the solution to the problem(16). ∎

A.5 Proof of Theorem 5

First, we make sure of a following basic result. Let us define a function $g(z_1, z_2) = {z_1}^{\alpha_1} {z_2}^{\alpha_2}$, $(z_1, z_2) \in \mathbb{R}_+^2$. Clearly, g is concave. This implies that for any $z = (z_1, z_2)$, $z' = (z_1', z_2') \in \mathbb{R}_+^2$ the inequality $g(z/2 + z'/2) \geqq \frac{1}{2}g(z) + \frac{1}{2}g(z')$ holds. That is,

$$\prod_{i=1}^{2} \left(\frac{z_i}{2} + \frac{z_i'}{2} \right)^{\alpha_i} \geqq \frac{1}{2} \prod_{i=1}^{2} {z_i}^{\alpha_i} + \frac{1}{2} \prod_{i=1}^{2} {z_i'}^{\alpha_i}. \tag{35}$$

We pick two vectors $X = (K, L), X' = (K', L')$ arbitrarily in $\mathbb{R}_{++}^2$. For symbolic simplicity, we write

$$K_i^x = K_i[X], \quad L_i^x = L_i[X], \quad K_i^{x\prime} = K_i[X'], \ L_i^{x\prime} = L_i[X'], \ i = 1, 2.$$

It is obvious that $\sum_{i=1}^2 (K_i^x + K_i^{x\prime}, L_i^x + L_i^{x\prime}) \leqq X + X'$. Then $(K_i^x + K_i^{x\prime}, L_i^x + L_i^{x\prime})_{i=1}^2 \in A(X + X')$. By (16), we know that

$$\prod_{i=1}^{2} \left(\frac{F_i(K_i[X + X'], L_i[X + X'])}{\alpha_i} \right)^{\alpha_i} \geqq \prod_{i=1}^{2} \left(\frac{F_i(K_i^x + K_i^{x\prime}, L_i^x + L_i^{x\prime})}{\alpha_i} \right)^{\alpha_i}. \tag{36}$$

Since $F_i(\cdot, \cdot)$ is homogenous of degree one and concave, then next inequality holds.

$$\begin{aligned} F_i(K_i^x + K_i^{x\prime}, L_i^x + L_i^{x\prime}) &= 2F_i(K_i^x/2 + K_i^{x\prime}/2, L_i^x/2 + L_i^{x\prime}/2) \\ &\geqq 2 \left(\frac{1}{2} F_i(K_i^x, L_i^x) + \frac{1}{2} F_i(K_i^{x\prime}, L_i^{x\prime}) \right). \end{aligned}$$

This together with (35) implies

$$\prod_{i=1}^{2}\left(\frac{F_i(K_i^x+K_i^{x\prime},L_i^x+L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i} \geqq 2\prod_{i=1}^{2}\left(\frac{1}{2}\frac{F_i(K_i^x,L_i^x)}{\alpha_i}+\frac{1}{2}\frac{F_i(K_i^{x\prime},L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i}$$
$$\geqq \prod_{i=1}^{2}\left(\frac{F_i\left(K_i^x,L_i^x\right)}{\alpha_i}\right)^{\alpha_i}+\prod_{i=1}^{2}\left(\frac{F_i(K_i^{x\prime},L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i}.$$

The above inequality and (36) imply

$$F(X+X') \geqq F(X)+F(X').$$

In addition to this F is homogenous of degree one. Then F is a concave function. ▮

A.6 Proof of Theorem 6

To distinguish two Uzawa-economies, we write Uzawa-economies $((K,L),(F_i)_{i=1}^2,(\alpha_i)_{i=1}^2)$ and $((\bar{K},\bar{L}),(F_i)_{i=1}^2,(\alpha_i)_{i=1}^2)$ as $\mathcal{U}_0$ and $\mathcal{U}_1$, respectively. First, we note that the capital labor ratios $k_i(\omega), i=1,2$ in $\mathcal{U}_1$ are identical with those in $\mathcal{U}_0$, since F_i's are common in two economies. Second, let us consider the equilibrium of commodity markets. By (3′) and the Euler's theorem on F_i in $\mathcal{U}_0$, it holds that

$$K_i(\omega)+\omega L_i(\omega)=p_i(\omega)f_i(k_i(\omega))L_i(\omega)=\alpha_i(K+\omega L), i=1,2.$$

Adding these up with respect to i leads us to $\bar{K}+\omega\bar{L}=K(\omega)+\omega L(\omega)=K+\omega L$. This fact leads us to

$$p_i(\omega)f_i(k_i(\omega))L_i(\omega)=\alpha_i(\bar{K}+\omega\bar{L}),\ i=1,2.$$

This implies that the demand for labor and capital of the i-th sector in $\mathcal{U}_1$ are identical with $L_i(\omega)$ and $K_i(\omega)$ in $\mathcal{U}_0$, $i=1,2$. And thus in $\mathcal{U}_1$ two factor markets are in balance at ω. Then the pair of price and allocation $(((p_i(\omega))_{i=1}^2,\omega),(Y_i\ (\omega),K_i(\omega),L_i(\omega))_{i=1}^2)$ is an equilibrium in $\mathcal{U}_1$. The second assertion holds obviously. ▮

A.7 Proof of Lemma 4

Let $Y_i[K,L],K_i[K,L],L_i[K,L],i=1,2$ be the solution to the problem (16). Of course, $Y_i[K,L]=F_i(K_i[K,L],L_i[K,L])$ is true, $i=1,2$. We express functions in this way by using blackbracketsets to distinguish those in Section 4.1 from those in Section 2. For the simplicity of expression, we write $p_i=p_i(\omega^*),i=1,2,\ p=(p_1)^{\alpha_1}(p_2)^{\alpha_2}$, $Y_i=Y_i[K,L],i=1,2$. From Theorems 2 and 3, mutual relations among variables are:

$$\lambda_i=\frac{p_i}{p},i=1,2,\ \mu=\frac{1}{p},\ \delta=\frac{\omega^*}{p},$$
$$Y_i=Y_i[K,L]=Y_i^*,\ K_i[K,L]=K_i^*,\ L_i[K,L]=L_i^*,i=1,2.$$

Let us show the marginal conditions $\partial F/\partial K = 1/p$ and $\partial F/\partial L = \omega^*/p$ to establish (aL). Our first step is (17), i.e.,

$$\alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i} = \frac{p_i}{p}.$$

Multiplying both sides of the above equation by $\partial F_i/\partial K_i(K_i^*, L_i^*)$ or $\partial F_i/\partial L_i(K_i^*, L_i^*)$, we obtain

$$\text{by (18),} \quad \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i}\frac{\partial F_i}{\partial K_i} = \frac{p_i}{p}\frac{\partial F_i}{\partial K_i} = \frac{1}{p}, \tag{37}$$

$$\text{by (19),} \quad \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i}\frac{\partial F_i}{\partial L_i} = \frac{p_i}{p}\frac{\partial F_i}{\partial L_i} = \frac{\omega^*}{p}. \tag{38}$$

On the other hand, equalities:

$$K_1[K,L] + K_2[K,L] = K, \;\; L_1[K,L] + L_2[K,L] = L$$

are identities. Then we obtain:

$$\frac{\partial K_1}{\partial K}[K,L] + \frac{\partial K_2}{\partial K}[K,L] = 1, \quad \frac{\partial L_1}{\partial K}[K,L] + \frac{\partial L_2}{\partial K}[K,L] = 0.$$

By these equations, we have

$$\sum_{i=1}^{2} \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i}\frac{\partial F_i}{\partial K_i}\frac{\partial K_i}{\partial K} = \frac{1}{p},$$

$$\sum_{i=1}^{2} \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i}\frac{\partial F_i}{\partial L_i}\frac{\partial L_i}{\partial K} = \frac{\omega^*}{p}\sum_{i=1}^{2}\frac{\partial L_i}{\partial K} = 0.$$

Adding up these two equation leads us to:

$$\begin{aligned}\frac{1}{p} &= \sum_{i=1}^{2} \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i}\left(\frac{\partial F_i}{\partial K_i}\frac{\partial K_i}{\partial K} + \frac{\partial F_i}{\partial L_i}\frac{\partial L_i}{\partial K}\right)\\ &= \sum_{i=1}^{2} \alpha_i \frac{\left(\frac{F_1(K_1(X),L_1(X))}{\alpha_1}\right)^{\alpha_1}\left(\frac{F_2(K_2(X),L_2(X))}{\alpha_2}\right)^{\alpha_2}}{\bar{Y}_i}\left(\frac{\partial F_i}{\partial K_i}\frac{\partial K_i}{\partial K} + \frac{\partial F_i}{\partial L_i}\frac{\partial L_i}{\partial K}\right)\\ &= \frac{\partial F}{\partial K}(K,L),\end{aligned} \tag{39}$$

where $X = (K, L)$. It may be a little bit repetitive, we show another marginal condition. Next equalities are also obvious.

$$\frac{\partial K_1}{\partial L}[K, L] + \frac{\partial K_2}{\partial L}[K, L] = 0, \quad \frac{\partial L_1}{\partial L}[K, L] + \frac{\partial L_2}{\partial L}[K, L] = 1.$$

This together with (37) and (38) implies

$$\begin{aligned}
\frac{\partial F}{\partial L}(K, L) &= \sum_{i=1}^{2} \alpha_i \frac{\left(\frac{Y_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{Y_2}{\alpha_2}\right)^{\alpha_2}}{Y_i} \left(\frac{\partial F_i}{\partial K_i}\frac{\partial K_i}{\partial L} + \frac{\partial F_i}{\partial L_i}\frac{\partial L_i}{\partial L}\right) \\
&= \sum_{i=1}^{2} \left(\frac{p_i}{p}\frac{\partial F_i}{\partial K_i}\frac{\partial K_i}{\partial L} + \frac{p_i}{p}\frac{\partial F_i}{\partial L_i}\frac{\partial L_i}{\partial L}\right) = \sum_{i=1}^{2} \left(\frac{1}{p}\frac{\partial K_i}{\partial L} + \frac{\omega^*}{p}\frac{\partial L_i}{\partial L}\right) \\
&= \frac{\omega^*}{p}. \qquad (40)
\end{aligned}$$

This is the second marginal condition for maximization problem in (aL). The pair (39) and (40) constitutes a sufficient condition for maximization problem in (aL), since F is a concave function. ∎